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NUMERICAL ANALYSIS OF A DUAL-MIXED PROBLEM IN NON-STANDARD BANACH SPACES*

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Abstract. In this paper we analyze the numerical approximation of a saddle-point problem posed in non-standard Banach spaces $\mathrm{H}(\mathrm{div}_p\,,\Omega)\times L^q(\Omega)$, where $\mathrm{H}(\mathrm{div}_p\,,\Omega):=\{\boldsymbol{\tau}\in[L^2(\Omega)]^n\colon \mathrm{div}\boldsymbol{\tau}\in L^p(\Omega)\}$, with $p>1$ and $q\in\mathbb{R}$ being the conjugate exponent of p and $\Omega\subseteq\mathbb{R}^n$ ($n\in\{2,3\}$) a bounded domain with Lipschitz boundary Γ. In particular, we are interested in deriving the stability properties of the forms involved (inf-sup conditions, boundedness), which are the main ingredients to analyze mixed formulations. In fact, by using these properties we prove the well-posedness of the corresponding continuous and discrete saddle-point problems by means of the classical Babuška-Brezzi theory, where the associated Galerkin scheme is defined by Raviart-Thomas elements of order $k\geq 0$ combined with piecewise polynomials of degree k. In addition we prove optimal convergence of the numerical approximation in the associated Lebesgue norms. Next, by employing the theory developed for the saddle-point problem, we analyze a mixed finite element method for a convection-diffusion problem, providing well-posedness of the continuous and discrete problems and optimal convergence under a smallness assumption on the convective vector field. Finally, we corroborate the theoretical results with suitable numerical results in two and three dimensions.

Key words. mixed finite element method, Raviart-Thomas, Lebesgue spaces, Lp data, convection-diffusion

AMS subject classifications. 65N15, 65N12, 65N30, 74S05

1. Introduction.

This paper is concerned with the solvability and numerical approximation of the saddle-point problem: Given $F\in[\mathrm{H}(\mathrm{div}_p\,,\Omega)]'$ and $G\in[L^q(\Omega)]'$, find $(\boldsymbol{\sigma},u)\in\mathrm{H}(\mathrm{div}_p\,,\Omega)\times L^q(\Omega)$ such that

$$\begin{aligned}\int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\tau}+\int_\Omega u\,\mathrm{div}\boldsymbol{\tau} &= F(\boldsymbol{\tau}) &&\forall\,\boldsymbol{\tau}\in\mathrm{H}(\mathrm{div}_p\,,\Omega),\\ \int_\Omega v\,\mathrm{div}\boldsymbol{\sigma} &= G(v) &&\forall\,v\in L^q(\Omega),\end{aligned}\tag{1.1}$$

where $\Omega\subseteq\mathbb{R}^n$ ($n=2,3$) is a bounded domain with Lipschitz boundary Γ, and given $p>\frac{2n}{n+2}$, $\mathrm{H}(\mathrm{div}_p\,,\Omega)$ is the Banach space defined as

$$\mathrm{H}(\mathrm{div}_p\,,\Omega):=\{\boldsymbol{\tau}\in[L^2(\Omega)]^n\colon \mathrm{div}\boldsymbol{\tau}\in L^p(\Omega)\}$$

endowed with the norm

$$\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathrm{div}_p\,,\Omega)}:=\left(\|\boldsymbol{\tau}\|^2_{L^2(\Omega)}+\|\mathrm{div}\boldsymbol{\tau}\|^2_{L^p(\Omega)}\right)^{1/2},$$

and $q\in\mathbb{R}$ is the conjugate exponent of p satisfying $\frac{1}{p}+\frac{1}{q}=1$. In particular, we are interested in providing the stability properties of the forms involved (inf-sup conditions, boundedness),

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at the continuous and discrete level, which are the main requirements to deduce the well-posedness of (1.1) and its Galerkin scheme and to derive the corresponding error analysis. We believe that, by having these stability properties, one can easily analyze mixed formulations of other interesting problems in fluid and solid mechanics. For instance, to approximate the flux of certain concentration θ in a flow-transport system with θ satisfying the convection-diffusion equation:

$$-\Delta\theta + \mathbf{v}\cdot\nabla\theta = g \quad \text{in} \quad \Omega, \qquad \theta = \theta_D \quad \text{on} \quad \Gamma, \tag{1.2}$$

where $\mathbf{v}$ is a given function in $[\mathrm{H}^1(\Omega)]^n$ representing the velocity of a viscous fluid occupying the region Ω where the concentration is moving and $g \in L^2(\Omega)$ and $\theta_D \in H^{1/2}(\Gamma)$ are given data, certainly the best option is to use a mixed method. To that end, we introduce the further unknown $\boldsymbol{\sigma} := \nabla\theta$ in Ω and apply a suitable integration by parts formula to arrive at the mixed variational formulation of (1.2): Find $\boldsymbol{\sigma}$ and θ in suitable spaces such that

$$\begin{aligned} \int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\tau} \;+\; \int_\Omega \theta\,\mathrm{div}\boldsymbol{\tau} \;&=\; \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, \theta_D\rangle_\Gamma\,, \\ \int_\Omega \psi\,\mathrm{div}\boldsymbol{\sigma} - \int_\Omega \psi(\mathbf{v}\cdot\boldsymbol{\sigma}) \;&=\; -\int_\Omega g\psi, \end{aligned} \tag{1.3}$$

for all $\boldsymbol{\tau}$ and ψ, where $\boldsymbol{\nu}$ is the unit outward normal to Ω and $\langle\cdot,\cdot\rangle_\Gamma$ is the duality pairing of $\mathrm{H}^{-1/2}(\Gamma)$ and $\mathrm{H}^{1/2}(\Gamma)$ with respect to the $L^2(\Gamma)$-inner product. Now, in order to define the spaces for the corresponding unknowns and test functions, we notice that the first term of the first equation of (1.3) is well defined if $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ are in $[L^2(\Omega)]^n$. However, if $\boldsymbol{\sigma} \in [L^2(\Omega)]^n$, then the second term of the second equation of (1.3) forces the test function ψ to live in a space smaller than $L^2(\Omega)$, and as a consequence, the term $\mathrm{div}\boldsymbol{\sigma}$ shall be in a larger space than $L^2(\Omega)$. Indeed, by applying Cauchy-Schwarz and Hölder inequalities and then the continuous injection of $\mathrm{H}^1(\Omega)$ into $L^4(\Omega)$ (see, e.g., [21, Theorem 1.3.4]), we obtain that there exists a constant $c(\Omega)$ such that

$$\begin{aligned} \left|\int_\Omega \psi(\mathbf{v}\cdot\boldsymbol{\sigma})\right| &\le \|\psi\mathbf{v}\|_{[L^2(\Omega)]^n}\|\boldsymbol{\sigma}\|_{[L^2(\Omega)]^n} \le \|\psi\|_{L^4(\Omega)}\|\mathbf{v}\|_{[L^4(\Omega)]^n}\|\boldsymbol{\sigma}\|_{[L^2(\Omega)]^n} \\ &\le c(\Omega)\|\psi\|_{L^4(\Omega)}\|\mathbf{v}\|_{[H^1(\Omega)]^n}\|\boldsymbol{\sigma}\|_{[L^2(\Omega)]^n}\,. \end{aligned} \tag{1.4}$$

According to the above, we obtain that the mixed problem (1.3) is well defined if the unknown θ and the test function ψ live both in $L^4(\Omega)$, whereas $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ live in $\mathrm{H}(\mathrm{div}_{4/3},\Omega)$, where

$$\mathrm{H}(\mathrm{div}_{4/3},\Omega) := \{\boldsymbol{\tau} \in [L^2(\Omega)]^n\colon \mathrm{div}\boldsymbol{\tau} \in L^{4/3}(\Omega)\}.$$

Observe that if $\mathbf{v} \in \mathrm{H}^1(\Omega) \cap L^\infty(\Omega)$, in (1.4) we could have bounded $\mathbf{v}$ in the L^∞-norm and keep ψ and $\mathrm{div}\boldsymbol{\sigma}$ in $L^2(\Omega)$. However, since (1.2) is usually coupled with an equation modeling the velocity $\mathbf{v}$, the estimate of $\mathbf{v}$ in the H^1-norm is required to analyze the full system. For instance, in [10, 11] (see also [5, 7, 15, 20]) to analyze the well-posedness of a new augmented mixed formulation for the Boussinesq model, which consists of a system of equations where the Navier-Stokes equation is coupled with a convection-diffusion equation of the type (1.2), a fixed-point strategy is proposed, where the estimate of the velocity in the convective term, at first in the L^4-norm and later in the H^1-norm, is crucial to achieve the Lipschitz continuity and the contractive property of the corresponding fixed-point operator. A similar approach can be found in [2] for a flow-transport problem. We also remark that if $\mathbf{v}$ is in $L^2(\Omega)$ or in $H(\mathrm{div})$, then the analysis below cannot be applied since the term $|\int_\Omega \psi(\mathbf{v}\cdot\boldsymbol{\sigma})|$ cannot be bounded properly. In particular, the latter prevents the use of our approach when the velocity is modeled by Darcy's law. We refer the reader to [3, 4] for this type of coupled models and observe that in both contributions the heat equation is written in the standard primal form.

A more simple problem that also fits in the framework of (1.1) is the Poisson problem with data $f \in L^p(\Omega)$, $p > \frac{2n}{n+2}$:

$$-\Delta u = f \quad \text{in} \quad \Omega \qquad \text{and} \qquad u = u_D \quad \text{on} \quad \Gamma. \tag{1.5}$$

In fact, by setting $\boldsymbol{\sigma} = \nabla u$ in Ω, it readily follows that $-\mathrm{div}\boldsymbol{\sigma} = f \in L^p(\Omega)$, and then, integrating by parts, one easily arrive at (1.1) with $F(\boldsymbol{\tau}) = \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}, u_D \rangle_\Gamma$ and $G(v) = -\int_\Omega f\, v$.

Now, concerning the error analysis of mixed finite element methods in Lebesgue spaces, among the few works available in the literature we could mention the paper [13] where the author focuses on deriving the a priori error estimate in L^p, with $1 \leq p \leq \infty$, of the numerical approximation of the standard $H(\mathrm{div}, \Omega) \times L^2(\Omega)$ dual-mixed formulation of the Poisson problem (1.5) in $\mathbb{R}^2$. Similarly, in [1] the authors focus on proving error estimates in L^p, with $1 \leq p \leq \infty$, for the 3D Raviart-Thomas approximation of mixed problems. We emphasize that in both papers the error analysis is derived by assuming that the unknowns $\boldsymbol{\sigma}$ and u are in the standard Hilbert spaces $H(\mathrm{div})$ and L^2, which differs from our approach.

According to the above discussion, in this paper we analyze the solvability and numerical approximation of the mixed variational formulation of problem (1.1) with $p > \frac{2n}{n+2}$. We employ the classical Babuška-Brezzi theory to study the well-posedness of the continuous problem. Since the Lebesgue and Sobolev spaces involved are not standard, the main drawback appears when proving the corresponding inf-sup condition, which is overcome by using suitable auxiliary problems. Similarly, we obtain that the associated Galerkin scheme, defined by Raviart-Thomas elements of order $k \geq 0$ and piecewise polynomials of degree k defined on a regular mesh, is well posed and convergent. Next, we apply the theory developed for the saddle-point problem (1.1) to analyze the convection-diffusion problem (1.3). More precisely, we combine the stability properties of the forms defining (1.1) with the well-known Banach-Nečas-Babuška theorem (cf. [14, Theorem 2.6]) and obtain that, under a smallness assumption on the coefficient $\mathbf{v}$ in the H^1-norm, problem (1.3) and its corresponding Galerkin scheme are well-posed.

The rest of the article is organized as follows. In Section 2 we prove the well-posedness of problem (1.1) by means of the classical Babuška-Brezzi theory. The corresponding Galerkin scheme is defined and analyzed in Section 3. Next, in Section 4 we apply the results derived in the previous sections to the convection-diffusion problem (1.2). Finally, several numerical results illustrating the performance of the mixed method are presented in Section 5.

We end this section by fixing some notation. Throughout the rest of the paper, we utilize the standard terminology for Lebesgue and Sobolev spaces, norms, and seminorms. In fact, let $\mathcal{O}$ be a domain in $\mathbb{R}^n$, $n = 2, 3$, with Lipschitz boundary $\partial\mathcal{O}$. For $r \geq 0$ and $p \in [1, \infty]$, we denote by $L^p(\mathcal{O})$ and $W^{r,p}(\mathcal{O})$ the usual Lebesgue and Sobolev spaces endowed with the norms $\|\cdot\|_{L^p(\mathcal{O})}$ and $\|\cdot\|_{W^{r,p}(\mathcal{O})}$, respectively. Note that $W^{0,p}(\mathcal{O}) = L^p(\mathcal{O})$. If $p = 2$, we write $\mathrm{H}^r(\mathcal{O})$ in place of $W^{r,2}(\mathcal{O})$ and denote the corresponding Lebesgue and Sobolev norms by $\|\cdot\|_{0,\mathcal{O}}$ and $\|\cdot\|_{r,\mathcal{O}}$, respectively. For $r \geq 0$, we write $|\cdot|_{r,\mathcal{O}}$ for the H^r-seminorm. The space $W_0^{1,p}(\mathcal{O})$ is the space of functions in $W^{1,p}(\mathcal{O})$ with vanishing trace on $\partial\mathcal{O}$. Also, the Hilbert space

$$\mathrm{H}(\mathrm{div}, \mathcal{O}) := \left\{ \boldsymbol{\tau} \in [L^2(\mathcal{O})]^n : \mathrm{div}\, \boldsymbol{\tau} \in L^2(\mathcal{O}) \right\},$$

is standard in the realm of mixed problems (see [6] or [18] for instance).

In what follows, we employ $\mathbf{0}$ to denote a generic null vector and use C and c, with or without subscripts, bars, tildes or hats, to denote generic positive constants independent of the discretization parameters, which may take different values at different places.

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Finally, given two Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, the norm in the product space $X \times Y$ will be denoted in the sequel by $\|(\cdot,\cdot)\|$ and will be defined as

$$\|(x, y)\| = \|x\|_X + \|y\|_Y .$$

2. Analysis of the continuous problem.

2.1. Preliminaries. Here, we introduce some notations and preliminary results that will serve for the forthcoming analysis. We begin by defining the sign function sgn, given by

$$\operatorname{sgn}(v) := \begin{cases} 1 & \text{if} \quad v \geq 0, \\ -1 & \text{if} \quad v < 0, \end{cases}$$

for any scalar function v. It is quite clear that for a given v, $v \operatorname{sgn}(v) = |v|$.

In the sequel we will make use of the well known Hölder, Poincaré, and Sobolev inequalities:

$$\int_\Omega |fg| \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}, \quad \forall\, f \in L^p(\Omega),\ \forall\, g \in L^q(\Omega), \quad \text{with} \quad \tfrac{1}{p} + \tfrac{1}{q} = 1, \tag{2.1}$$

$$\|w\|_{1,\Omega} \leq C_P |w|_{1,\Omega} \quad \forall w \in \mathrm{H}^1_0(\Omega), \tag{2.2}$$

and

$$\|w\|_{L^r(\Omega)} \leq C_{Sob} \|w\|_{1,\Omega} \quad \forall\, w \in \mathrm{H}^1(\Omega)\,, \quad \begin{cases} r \geq 1, & \text{if } n = 2, \\ r \in [1, 6] & \text{if } n = 3, \end{cases} \tag{2.3}$$

with $C_P > 0$ and $C_{Sob} > 0$ depending only on $|\Omega|$.

2.2. Well-posedness. In what follows we address the existence and uniqueness of a solution of problem (1.1). To that end, and for the sake of simplicity, we now write our problem in the classical variational setting and state the main properties of the bilinear forms involved. We start by defining the spaces

$$\mathrm{H} := \mathrm{H}(\operatorname{div}_p, \Omega) \qquad \text{and} \qquad \mathrm{Q} := L^q(\Omega).$$

Then, defining the bilinear forms $a : \mathrm{H} \times \mathrm{H} \to \mathbb{R}$ and $b : \mathrm{H} \times \mathrm{Q} \to \mathbb{R}$ by

$$a(\boldsymbol{\sigma}, \boldsymbol{\tau}) := \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau} \qquad \text{and} \qquad b(\boldsymbol{\tau}, v) := \int_\Omega v \operatorname{div} \boldsymbol{\tau}, \tag{2.4}$$

the variational formulation (1.1) reads: Find $(\boldsymbol{\sigma}, u) \in \mathrm{H} \times \mathrm{Q}$ such that

$$\begin{aligned} a(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, u) &= F(\boldsymbol{\tau}) && \forall\, \boldsymbol{\tau} \in \mathrm{H}, \\ b(\boldsymbol{\sigma}, v) &= G(v) && \forall\, v \in \mathrm{Q}. \end{aligned} \tag{2.5}$$

Notice that, owing to the Hölder inequality (2.1), the bilinear forms a and b are bounded:

$$\begin{aligned} |a(\boldsymbol{\sigma}, \boldsymbol{\tau})| &\leq \|\boldsymbol{\sigma}\|_{\mathrm{H}} \|\boldsymbol{\tau}\|_{\mathrm{H}} && \forall\, \boldsymbol{\sigma} \in \mathrm{H},\ \forall\, \boldsymbol{\tau} \in \mathrm{H}, \\ |b(\boldsymbol{\tau}, v)| &\leq \|v\|_{\mathrm{Q}} \|\boldsymbol{\tau}\|_{\mathrm{H}} && \forall\, \boldsymbol{\tau} \in \mathrm{H},\ \forall\, v \in \mathrm{Q}. \end{aligned}$$

Throughout the rest of this section we employ the classical Babuška-Brezzi theory in Banach spaces (e.g., [14, Theorem 2.34]) to conclude that (2.5) is well posed. This requires an inf-sup condition of b and two inf-sup conditions of a on the kernel of b. We start with the inf-sup condition of b.

LEMMA 2.1. *There exists $\beta > 0$ such that*

$$\sup_{\boldsymbol{\tau}\in \mathrm{H}\setminus\mathbf{0}} \frac{b(\boldsymbol{\tau},v)}{\|\boldsymbol{\tau}\|_{\mathrm{H}}} \geq \beta \|v\|_{\mathrm{Q}} \qquad \forall\, v\in \mathrm{Q}. \tag{2.6}$$

Proof. Given $v\in \mathrm{Q}$, we let $\tilde{\boldsymbol{\tau}} = -\nabla z$, with $z\in \mathrm{H}^1_0(\Omega)$ being the unique solution of the variational problem

$$\int_\Omega \nabla z\cdot\nabla w = \int_\Omega \operatorname{sgn}(v)|v|^{q-1} w \qquad \forall\, w\in \mathrm{H}^1_0(\Omega). \tag{2.7}$$

Notice that

$$\int_\Omega \left|\operatorname{sgn}(v)|v|^{q-1}\right|^p = \int_\Omega |v|^{p(q-1)} = \int_\Omega |v|^q < +\infty,$$

which implies that $\operatorname{sgn}(v)|v|^{q-1}\in L^p(\Omega)$. Then, since $p > \frac{2n}{n+2}$, it is well-known that problem (2.7) is well posed. In turn, from (2.7) it readily follows that $\operatorname{div}\tilde{\boldsymbol{\tau}} = \operatorname{sgn}(v)|v|^{q-1}$. As a consequence, we obtain that $\tilde{\boldsymbol{\tau}}\in \mathrm{H}$ and

$$\|\operatorname{div}\tilde{\boldsymbol{\tau}}\|_{L^p(\Omega)} = \||v|^{q-1}\|_{L^p(\Omega)}. \tag{2.8}$$

On the other hand, utilizing inequalities (2.1), (2.2), and (2.3), from (2.7) with $w = z$ we obtain

$$\begin{aligned}\|\tilde{\boldsymbol{\tau}}\|^2_{0,\Omega} &\leq \||v|^{q-1}\|_{L^p(\Omega)}\|z\|_{\mathrm{Q}} \leq C_{Sob}\||v|^{q-1}\|_{L^p(\Omega)}\|z\|_{1,\Omega}\\ &\leq C_P C_{Sob}\||v|^{q-1}\|_{L^p(\Omega)}|z|_{1,\Omega} = C_P C_{Sob}\||v|^{q-1}\|_{L^p(\Omega)}\|\tilde{\boldsymbol{\tau}}\|_{0,\Omega},\end{aligned}$$

from which,

$$\|\tilde{\boldsymbol{\tau}}\|_{0,\Omega} \leq C_P C_{Sob}\||v|^{q-1}\|_{L^p(\Omega)}. \tag{2.9}$$

In this way, from (2.8) and (2.9) we have

$$\|\tilde{\boldsymbol{\tau}}\|_{\mathrm{H}} \leq (1 + C_P^2 C_{Sob}^2)^{1/2}\||v|^{q-1}\|_{L^p(\Omega)},$$

which together with the fact that

$$\||v|^{q-1}\|_{L^p(\Omega)} = \left(\int_\Omega (|v|^{q-1})^p\right)^{\frac{1}{p}} = \left(\int_\Omega |v|^q\right)^{\frac{q-1}{q}} = \|v\|_Q^{q-1}$$

implies

$$\|\tilde{\boldsymbol{\tau}}\|_{\mathrm{H}} \leq (1 + C_P^2 C_{Sob}^2)^{1/2}\|v\|_{\mathrm{Q}}^{q-1}. \tag{2.10}$$

Therefore, recalling that $v\operatorname{sgn}(v) = |v|$, from the definition of $\tilde{\boldsymbol{\tau}}$ and (2.10), we obtain

$$\begin{aligned}\sup_{\boldsymbol{\tau}\in \mathrm{H}\setminus\mathbf{0}} \frac{b(\boldsymbol{\tau},v)}{\|\boldsymbol{\tau}\|_{\mathrm{H}}} &\geq \frac{b(\tilde{\boldsymbol{\tau}},v)}{\|\tilde{\boldsymbol{\tau}}\|_{\mathrm{H}}} = \frac{\displaystyle\int_\Omega v\operatorname{div}\tilde{\boldsymbol{\tau}}}{\|\tilde{\boldsymbol{\tau}}\|_{\mathrm{H}}} \geq (1 + C_P^2 C_{Sob}^2)^{-1/2}\frac{\displaystyle\int_\Omega |v|\,|v|^{q-1}}{\|v\|_{\mathrm{Q}}^{q-1}}\\ &= (1 + C_P^2 C_{Sob}^2)^{-1/2}\frac{\|v\|_{\mathrm{Q}}^q}{\|v\|_{\mathrm{Q}}^{q-1}} = (1 + C_P^2 C_{Sob}^2)^{-1/2}\|v\|_{\mathrm{Q}},\end{aligned}$$

which concludes the proof with $\beta = (1 + C_P^2 C_{Sob}^2)^{-1/2} > 0$. $\square$

We now let V be the kernel of b, that is

$$V := \{\boldsymbol{\tau} \in \mathrm{H} \colon b(\boldsymbol{\tau}, v) = 0,\ \forall\, v \in \mathrm{Q}\} = \left\{\boldsymbol{\tau} \in \mathrm{H} \colon \int_\Omega v \operatorname{div} \boldsymbol{\tau} = 0,\ \forall\, v \in \mathrm{Q}\right\}.$$

Observe that if $\boldsymbol{\tau} \in V$, then taking $v = \operatorname{sgn}(\operatorname{div}\boldsymbol{\tau})|\operatorname{div}\boldsymbol{\tau}|^{p-1}$, which is clearly an element in Q since

$$\int_\Omega |v|^q = \int_\Omega \left|\operatorname{sgn}(\operatorname{div}\boldsymbol{\tau})|\operatorname{div}\boldsymbol{\tau}|^{p-1}\right|^q = \int_\Omega |\operatorname{div}\boldsymbol{\tau}|^{(p-1)q} = \int_\Omega |\operatorname{div}\boldsymbol{\tau}|^p < +\infty,$$

it follows that

$$0 = \int_\Omega v \operatorname{div}\boldsymbol{\tau} = \int_\Omega \operatorname{sgn}(\operatorname{div}\boldsymbol{\tau})|\operatorname{div}\boldsymbol{\tau}|^{p-1}\operatorname{div}\boldsymbol{\tau} = \int_\Omega |\operatorname{div}\boldsymbol{\tau}|^p = \|\operatorname{div}\boldsymbol{\tau}\|^p_{L^p(\Omega)},$$

and then $\operatorname{div}\boldsymbol{\tau} \equiv 0$ in $L^p(\Omega)$. In this way,

$$V := \{\boldsymbol{\tau} \in \ \mathrm{H} : \ \operatorname{div}\boldsymbol{\tau} \equiv 0 \quad \text{in } \Omega\}.$$

The following lemma establishes the corresponding inf-sup conditions of a on V.

LEMMA 2.2. *There exists $\alpha > 0$ such that*

$$\sup_{\boldsymbol{\tau} \in V\setminus\{\mathbf{0}\}} \frac{a(\boldsymbol{\zeta}, \boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\mathrm{H}}} \geq \alpha \|\boldsymbol{\tau}\|_{\mathrm{H}} \qquad \forall\, \boldsymbol{\zeta} \in V. \tag{2.11}$$

In addition,

$$\sup_{\zeta \in V} a(\boldsymbol{\zeta}, \boldsymbol{\tau}) > 0 \qquad \forall\, \boldsymbol{\tau} \in V\setminus\{\mathbf{0}\}. \tag{2.12}$$

Proof. Given $\boldsymbol{\tau} \in V$, from the definition of V, it readily follows that

$$a(\boldsymbol{\tau}, \boldsymbol{\tau}) = \|\boldsymbol{\tau}\|^2_{0,\Omega} = \|\boldsymbol{\tau}\|^2_{\mathrm{H}},$$

which clearly implies (2.11) with $\alpha = 1$. Moreover, the estimate (2.12) follows from (2.11) and the fact that $a(\boldsymbol{\zeta}, \boldsymbol{\tau}) = a(\boldsymbol{\tau}, \boldsymbol{\zeta})$ for all $\boldsymbol{\zeta}, \boldsymbol{\tau} \in \mathrm{H}$. ∎

The well-posedness of the continuous formulation (2.5) is provided now.

THEOREM 2.3. *Let $p > \frac{2n}{n+2}$, $F \in \mathrm{H}'$, and $G \in \mathrm{Q}'$. Then there exists a unique solution $(\boldsymbol{\sigma}, u) \in \mathrm{H} \times \mathrm{Q}$ to (2.5). In addition, there exists $C > 0$, independent of the solution, such that*

$$\|(\boldsymbol{\sigma}, u)\| \leq C(\|F\|_{\mathrm{H}'} + \|G\|_{\mathrm{Q}'}).$$

Proof. Thanks to Lemmas 2.1 and 2.2, the proof follows from a straightforward application of the Babuška-Brezzi theory in Banach spaces (see, e.g., [14, Theorem 2.34]). ∎

3. The mixed finite element scheme.

3.1. Preliminaries. Let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangulations of Ω by triangles T in $\mathbb{R}^2$ or tetrahedra in $\mathbb{R}^3$ of diameter h_T such that $h := \max\{h_T \colon T \in \mathcal{T}_h\}$. Let us recall that $\{\mathcal{T}_h\}_{h>0}$ is said to be regular if there exists $c > 0$ such that

$$\frac{h_T}{\rho_T} \leq c, \qquad \forall\, T \in \mathcal{T}_h \quad \forall\, h > 0, \tag{3.1}$$

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where $\rho_T > 0$ is the diameter of the largest circle or sphere contained in T. Then, for each $T \in \mathcal{T}_h$, we let $\mathrm{RT}_k(T)$ be the local Raviart-Thomas element of order $k \geq 0$, i.e.,

$$\mathrm{RT}_k(T) := [\mathbb{P}_k(T)]^n \oplus \mathbb{P}_k(T)\mathbf{x},$$

where $\mathbf{x} := (x_1, \ldots, x_n)^t$ is a generic vector of $\mathbb{R}^n$ and $\mathbb{P}_k(T)$ is the space of polynomials defined on T of degree $\leq k$. Hence, we define the following finite element subspaces to approximate the unknowns $\boldsymbol{\sigma} \in \mathrm{H}$ and $u \in \mathrm{Q}$:

$$\begin{aligned} \mathrm{H}_h &:= \{\boldsymbol{\tau}_h \in \mathrm{H} \colon \boldsymbol{\tau}_h|_T \in \mathrm{RT}_k(T), \quad \forall\, T \in \mathcal{T}_h\} \subseteq \mathrm{H}, \\ \mathrm{Q}_h &:= \{v_h \in \mathrm{Q} \colon v_h|_T \in \mathbb{P}_k(T), \quad \forall\, T \in \mathcal{T}_h\} \subseteq \mathrm{Q}. \end{aligned} \tag{3.2}$$

Then the conforming Galerkin scheme for (2.5) reads: Find $(\boldsymbol{\sigma}_h, u_h) \in \mathrm{H}_h \times \mathrm{Q}_h$ such that

$$\begin{aligned} a(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, u_h) &= F(\boldsymbol{\tau}_h) \qquad \forall\, \boldsymbol{\tau}_h \in \mathrm{H}_h, \\ b(\boldsymbol{\sigma}_h, v_h) &= G(v_h) \qquad \forall\, v_h \in \mathrm{Q}_h, \end{aligned} \tag{3.3}$$

where a and b are the bilinear forms defined in (2.4).

Next, in Section 3.2 we proceed similarly as in [17, Section 4.2] and employ the discrete Babuška-Brezzi theory to prove that problem (3.3) is well posed. To that end, we first need to establish some preliminary results and definitions. We begin by introducing the approximation properties of the finite element subspaces introduced above. To do that we first define the space

$$Z_p := \{\boldsymbol{\tau} \in \mathrm{H}(\mathrm{div}_p, \Omega) \colon \boldsymbol{\tau}|_T \in [W^{1,p}(T)]^n, \quad \forall\, T \in \mathcal{T}_h\},$$

and let

$$\Pi_h^k \colon Z_p \to \mathrm{H}_h,$$

be the Raviart-Thomas interpolation operator, which is well defined in Z_p (see, e.g., [14, Section 1.2.7]) and is characterized by the identities

$$\int_e (\Pi_h^k(\boldsymbol{\tau}) \cdot \boldsymbol{\nu})\, \xi = \int_e (\boldsymbol{\tau} \cdot \boldsymbol{\nu})\, \xi \qquad \forall\, \xi \in \mathbb{P}_k(e), \ \forall \text{ edge or face } e \text{ of } \mathcal{T}_h,$$

and

$$\int_T \Pi_h^k(\boldsymbol{\tau}) \cdot \boldsymbol{\psi} = \int_T \boldsymbol{\tau} \cdot \boldsymbol{\psi} \qquad \forall\, \boldsymbol{\psi} \in \mathbb{P}_{k-1}(T), \ \forall\, T \in \mathcal{T}_h \ (\text{if } k \geq 1)\,.$$

In addition, it is well known (see, e.g., [14, Lemma 1.41]) that the following identity holds

$$\mathrm{div}(\Pi_h^k(\boldsymbol{\tau})) = \mathcal{P}_h(\mathrm{div}\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in Z_p, \tag{3.4}$$

where $\mathcal{P}_h \colon L^2(\Omega) \to \mathrm{Q}_h$ is the corresponding orthogonal projection, which satisfies the following error estimate (see [14, Section 1.6.3]): For each $0 \leq t \leq k+1$ and for each $w \in W^{t,r}(\Omega)$, with $1 \leq r \leq \infty$, it holds that

$$\|w - \mathcal{P}_h(w)\|_{L^r(\Omega)} \leq C h^t |w|_{W^{t,r}(\Omega)}. \tag{3.5}$$

The following lemma establishes the local approximation properties of Π_h^k.

LEMMA 3.1. *Let $r > \frac{2n}{n+2}$. Then there exists $C_1 > 0$ independent of h such that for each $\boldsymbol{\tau} \in [W^{l+1,r}(T)]^n$, with $0 \leq l \leq k$, and for each $0 \leq m \leq l+1$,*

$$|\boldsymbol{\tau} - \Pi_h^k(\boldsymbol{\tau})|_{[W^{m,r}(T)]^n} \leq C_1 \frac{h_T^{l+2}}{\rho_T^{m+1}} |\boldsymbol{\tau}|_{[W^{l+1,r}(T)]^n}. \tag{3.6}$$

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Moreover, there exists $C_2 > 0$ independent of h such that for each $\boldsymbol{\tau} \in [W^{1,r}(T)]^n$, with $\mathrm{div}\boldsymbol{\tau} \in W^{l+1,r}(T)$ *and* $0 \le l \le k$, *and for each* $0 \le m \le l+1$,

$$|\mathrm{div}\boldsymbol{\tau} - \mathrm{div}(\Pi_h^k(\boldsymbol{\tau}))|_{W^{m,r}(T)} \le C_2 \frac{h_T^{l+1}}{\rho_T^m} |\mathrm{div}\boldsymbol{\tau}|_{W^{l+1,r}(T)}. \tag{3.7}$$

Proof. Employing the L^p-version of the Deny-Lions Lemma provided in [14, Lemma B.67] and the local estimates given in [14, Lemma 1.101], one can proceed analogously as in [17, Section 3.4.4] and prove that for any $r > \frac{2n}{n+2}$ the estimates (3.6) and (3.7) hold. We omit further details. □

Owing to the regularity of the mesh (cf. (3.1)) and from the estimates (3.6) and (3.7), it is not difficult to see that the following global estimate holds

$$\begin{aligned} &\|\boldsymbol{\tau} - \Pi_h^k(\boldsymbol{\tau})\|_{0,\Omega} + \|\mathrm{div}\boldsymbol{\tau} - \mathrm{div}(\Pi_h(\boldsymbol{\tau}))\|_{L^p(\Omega)} \\ &\qquad \le c h^{l+1} \Big\{ |\boldsymbol{\tau}|_{[H^{l+1}(\Omega)]^n} + |\mathrm{div}\boldsymbol{\tau}|_{W^{l+1,p}(\Omega)} \Big\}, \end{aligned} \tag{3.8}$$

for all $0 \le l \le k+1$ and for all $\boldsymbol{\tau} \in [H^{l+1}(\Omega)]^n$ with $\mathrm{div}\boldsymbol{\tau} \in W^{l+1,p}(\Omega)$.

REMARK 3.2. Notice that from the regularity of the mesh (cf. (3.1)) and from (3.6) with $m = 0$ and $m = 1$, one can easily obtain, respectively, that

$$\|\boldsymbol{\tau} - \Pi_h^k(\boldsymbol{\tau})\|_{[L^r(T)]^n} \le C_1 \frac{h_T^{l+2}}{\rho_T} |\boldsymbol{\tau}|_{[W^{l+1,r}(T)]^n} \le \hat{C}_1 h_T^{l+1} |\boldsymbol{\tau}|_{[W^{l+1,r}(T)]^n}$$

and

$$|\boldsymbol{\tau} - \Pi_h^k(\boldsymbol{\tau})|_{[W^{1,r}(T)]^n} \le C_2 \frac{h_T^{l+2}}{\rho_T^2} |\boldsymbol{\tau}|_{[W^{l+1,r}(T)]^n} \le \hat{C}_2 h_T^{l} |\boldsymbol{\tau}|_{[W^{l+1,r}(T)]^n},$$

which combined yields

$$\|\boldsymbol{\tau} - \Pi_h^k(\boldsymbol{\tau})\|_{[W^{1,r}(\Omega)]^n} \le C h^l |\boldsymbol{\tau}|_{[W^{l+1,r}(\Omega)]^n} \quad \forall\, \boldsymbol{\tau} \in [W^{l+1,r}(\Omega)]^n. \tag{3.9}$$

The latter will be employed below in the proof of Lemma 3.3.

3.2. Analysis of the discrete problem. In this section we apply the discrete Babuška-Brezzi theory to prove the well-posedness of the Galerkin scheme (3.3). We start by establishing the discrete inf-sup condition for b.

LEMMA 3.3. *There exists $\beta^* > 0$ independent of h such that*

$$\sup_{\substack{\boldsymbol{\tau}_h \in \mathrm{H}_h \\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{b(\boldsymbol{\tau}_h, v_h)}{\|\boldsymbol{\tau}_h\|_{\mathrm{H}}} \ge \beta^* \|v_h\|_{\mathrm{Q}} \qquad \forall\, v_h \in \mathrm{Q}_h. \tag{3.10}$$

Proof. Given $v_h \in \mathrm{Q}_h$, we set

$$\tilde{v}_h := \begin{cases} \mathrm{sgn}\,(v_h)|v_h|^{q-1} & \text{in} \quad \Omega, \\ 0 & \text{in} \quad B\backslash\overline{\Omega}, \end{cases}$$

where $B \subseteq \mathbb{R}^n$ is an open and bounded convex set containing $\overline{\Omega}$. Since $v_h \in L^p(\Omega)$, a well-known result on regularity of elliptic problems (see, e.g., [16]) implies that there exists a unique weak solution $\varphi \in W^{2,p}(B) \cap W_0^{1,p}(B)$ of the boundary value problem

$$-\Delta\varphi = \tilde{v}_h \quad \text{in} \quad B \qquad \text{and} \qquad \varphi = 0 \quad \text{on} \quad \partial B,$$

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which satisfies

$$\|\varphi\|_{W^{2,p}(\Omega)} \leq C\|\tilde{v}_h\|_{L^p(B)} = C\||v_h|^{q-1}\|_{L^p(\Omega)} = C\|v_h\|_Q^{q-1} \tag{3.11}$$

with $C > 0$. Hence, we set $\hat{\boldsymbol{\tau}} = -\nabla\varphi|_\Omega \in [W^{1,p}(\Omega)]^n$ and observe from (3.11) that

$$\|\hat{\boldsymbol{\tau}}\|_{[W^{1,p}(\Omega)]^n} \leq C\|v_h\|_Q^{q-1}, \tag{3.12}$$

which together with the continuous embedding from $W^{1,p}(\Omega)$ into $L^2(\Omega)$ implies

$$\|\hat{\boldsymbol{\tau}}\|_{0,\Omega} \leq C\|v_h\|_Q^{q-1}. \tag{3.13}$$

Now, we let $\hat{\boldsymbol{\tau}}_h = \Pi_h^k(\hat{\boldsymbol{\tau}})$ and observe from (3.4) that

$$\mathrm{div}\hat{\boldsymbol{\tau}}_h = \mathcal{P}_h(\mathrm{div}\hat{\boldsymbol{\tau}}) = \mathcal{P}_h(\mathrm{sgn}\,(v_h)|v_h|^{q-1}). \tag{3.14}$$

In turn, utilizing the triangle inequality, the continuous embedding from $W^{1,p}(\Omega)$ into $L^2(\Omega)$, and the estimate (3.13), we obtain

$$\|\hat{\boldsymbol{\tau}}_h\|_{0,\Omega} \leq \|\hat{\boldsymbol{\tau}} - \hat{\boldsymbol{\tau}}_h\|_{0,\Omega} + \|\hat{\boldsymbol{\tau}}\|_{0,\Omega} \leq c_1\|\hat{\boldsymbol{\tau}} - \hat{\boldsymbol{\tau}}_h\|_{[W^{1,p}(\Omega)]^n} + c_2\|v_h\|_Q^{q-1},$$

which together with (3.9) with $r = p$ and $l = 0$, and (3.12), imply

$$\|\hat{\boldsymbol{\tau}}_h\|_{0,\Omega} \;\leq\; C\|v_h\|_Q^{q-1}. \tag{3.15}$$

Hence, using the fact that $\mathcal{P}_h$ is a continuous operator, from (3.14) and (3.15), we easily obtain

$$\|\hat{\boldsymbol{\tau}}_h\|_{\mathrm{H}} = \left\{\|\hat{\boldsymbol{\tau}}_h\|_{0,\Omega}^2 + \|\mathrm{div}(\hat{\boldsymbol{\tau}}_h)\|_{L^p(\Omega)}^2\right\}^{1/2} \leq \hat{c}\|v_h\|_{\mathrm{Q}}^{q-1}, \tag{3.16}$$

with $\hat{c} > 0$ independent of h. Therefore, from (3.14) and (3.16), we find

$$\sup_{\substack{\boldsymbol{\tau}_h \in \mathrm{H}_h\\ \boldsymbol{\tau}_h \neq \theta}} \frac{b(\boldsymbol{\tau}_h, v_h)}{\|\boldsymbol{\tau}_h\|_{\mathrm{H}}} \geq \frac{b(\hat{\boldsymbol{\tau}}_h, v_h)}{\|\hat{\boldsymbol{\tau}}_h\|_{\mathrm{H}}} \geq \frac{1}{\hat{c}} \frac{\displaystyle\int_\Omega v_h\,\mathrm{sgn}\,(v_h)|v_h|^{q-1}}{\|v_h\|_{\mathrm{Q}}^{q-1}} = \frac{1}{\hat{c}}\frac{\|v_h\|_{\mathrm{Q}}^q}{\|v_h\|_{\mathrm{Q}}^{q-1}} = \frac{1}{\hat{c}}\|v_h\|_{\mathrm{Q}},$$

which concludes the proof with $\beta^* = \frac{1}{\hat{c}}$. ∎

We now look at the discrete kernel of b defined by

$$V_h := \{\boldsymbol{\tau}_h \in \mathrm{H}_h \,:\, b(\boldsymbol{\tau}_h, v_h) = 0,\ \forall\, v_h \in \mathrm{Q}_h\}\,.$$

Since $\mathrm{div}\mathrm{H}_h \subseteq \mathrm{Q}_h$, it readily follows that

$$V_h = \{\boldsymbol{\tau}_h \in \mathrm{H}_h \colon \mathrm{div}\boldsymbol{\tau}_h = 0 \quad \text{in} \quad \Omega\}\,.$$

The discrete version of Lemma 2.2 is shown next.

LEMMA 3.4. *There exists $\alpha^* > 0$ independent of h such that*

$$\sup_{\boldsymbol{\tau}_h \in V_h} \frac{a(\boldsymbol{\zeta}_h, \boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{\mathrm{H}}} \geq \alpha^*\|\boldsymbol{\zeta}_h\|_{\mathrm{H}} \qquad \forall\, \boldsymbol{\tau}_h \in V_h. \tag{3.17}$$

Proof. According to the definition of V_h, the proof follows analogously to the proof of the estimate (2.11) with $\alpha^* = 1$. ∎

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We recall here that in finite-dimensional spaces the discrete versions of (2.11) and (2.12) are equivalent, which is the reason why we only prove (3.17).

Owing to Lemmas 3.3 and 3.4, we are now in the position of establishing the solvability and stability of the Galerkin scheme (3.3) and the corresponding a priori error estimate.

THEOREM 3.5. *Let $p > \frac{2n}{n+2}$, $F \in \mathrm{H}'$, and $G \in \mathrm{Q}'$. Then there exists a unique solution $(\boldsymbol{\sigma}_h, u_h) \in \mathrm{H}_h \times \mathrm{Q}_h$ to (3.3). In addition, there exist C_1, $C_2 > 0$ independent of h such that*

$$\|(\boldsymbol{\sigma}_h, u_h)\| \leq C_1 \big\{ \|F|_{\mathrm{H}_h}\|_{\mathrm{H}_h'} + \|G|_{\mathrm{Q}_h}\|_{\mathrm{Q}_h'} \big\}$$

and

$$\|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, u - u_h)\| \leq C_2 \left\{ \inf_{\boldsymbol{\tau}_h \in \mathrm{H}_h} \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathrm{H}} + \inf_{v_h \in \mathrm{Q}_h} \|u - v_h\|_{\mathrm{Q}} \right\}, \tag{3.18}$$

where $(\boldsymbol{\sigma}, u) \in \mathrm{H} \times \mathrm{Q}$ is the unique solution of (2.5).

Proof. It follows from Lemmas 3.3 and 3.4 and a direct application of the discrete Babuška-Brezzi theory. □

We now provide the rate of convergence of our mixed finite element method.

THEOREM 3.6. *Let $p > \frac{2n}{n+2}$, and let $(\boldsymbol{\sigma}, u) \in \mathrm{H} \times \mathrm{Q}$ and $(\boldsymbol{\sigma}_h, u_h) \in \mathrm{H}_h \times \mathrm{Q}_h$ be the unique solutions of the continuous and discrete mixed formulations (2.5) and (3.3), respectively. Assume that $\boldsymbol{\sigma} \in [\mathrm{H}^{l+1}(\Omega)]^n$, $\mathrm{div}\boldsymbol{\sigma} \in W^{l+1,p}(\Omega)$, and $u \in W^{l+1,q}(\Omega)$, with $0 \leq l \leq k$. Then there exists $C > 0$ independent of h such that*

$$\|(\boldsymbol{\sigma}, u) - (\boldsymbol{\sigma}_h, u_h)\| \leq C h^{l+1} \left\{ \|\boldsymbol{\sigma}\|_{l+1,\Omega} + \|\mathrm{div}\boldsymbol{\sigma}\|_{W^{l+1,p}(\Omega)} + \|u\|_{W^{l+1,q}(\Omega)} \right\}. \tag{3.19}$$

Proof. From the approximation property (3.5) with $r = q$ and $t = l + 1$, we easily obtain

$$\|u - \mathcal{P}_h(u)\|_{\mathrm{Q}} \leq C h^{l+1} \|u\|_{W^{l+1,q}(\Omega)}. \tag{3.20}$$

Then, (3.19) readily follows from (3.8), (3.20), and the Céa estimate (3.18). □

4. Analysis of a convection-diffusion problem. In this section we address the unique solvability and numerical approximation of the convection-diffusion problem (1.3). To that end we let $d : \mathrm{H}(\mathrm{div}_{4/3}, \Omega) \times L^4(\Omega) \to \mathbb{R}$ be the bilinear form

$$d(\boldsymbol{\tau}, \psi) := -\int_\Omega (\mathbf{v} \cdot \boldsymbol{\tau})\psi,$$

and $F : \mathrm{H}(\mathrm{div}_{4/3}, \Omega) \to \mathbb{R}$ and $G : L^4(\Omega) \to \mathbb{R}$ be the functionals

$$F(\boldsymbol{\tau}) := \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}, \theta_D \rangle_\Gamma \quad \text{and} \quad G(\psi) := -\int_\Omega g\psi. \tag{4.1}$$

Then it is clear that the mixed variational problem (1.3) reads as follows: Find $(\boldsymbol{\sigma}, \theta) \in \mathrm{H}(\mathrm{div}_{4/3}, \Omega) \times L^4(\Omega)$ such that

$$\begin{aligned} a(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \theta) &= F(\boldsymbol{\tau}), \\ b(\boldsymbol{\sigma}, \psi) + d(\boldsymbol{\sigma}, \psi) &= G(\psi), \end{aligned} \tag{4.2}$$

for all $(\boldsymbol{\tau}, \psi) \in \mathrm{H}(\mathrm{div}_{4/3}, \Omega) \times L^4(\Omega)$, where a and b are the bilinear forms defined in (2.4). The unique solvability of (4.2) is derived in the next section.

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4.1. Analysis of the continuous problem. We begin this section by proving that F and G (cf. (4.1)) are well defined and bounded. Let us first recall that given $\boldsymbol{\tau} \in \mathrm{H}(\mathrm{div}, \Omega)$, the normal trace $\boldsymbol{\tau} \cdot \boldsymbol{\nu}$ is defined as the functional in $\mathrm{H}^{-1/2}(\Gamma)$ given by (see, e.g., [17, Section 1.3.4])

$$\langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}, \xi \rangle_\Gamma = \int_\Omega \boldsymbol{\tau} \cdot \nabla \tilde{\gamma}_0^{-1}(\xi) + \int_\Omega \tilde{\gamma}_0^{-1}(\xi)\, \mathrm{div} \boldsymbol{\tau} \qquad \forall\, \xi \in \mathrm{H}^{1/2}(\Gamma), \tag{4.3}$$

where $\tilde{\gamma}_0^{-1} \colon \mathrm{H}^{1/2}(\Gamma) \to [\mathrm{H}_0^1(\Omega)]^\perp$ is the right inverse of the well-known trace operator $\gamma_0 \colon \mathrm{H}^1(\Omega) \to \mathrm{H}^{1/2}(\Gamma)$. Then, since $\tilde{\gamma}_0^{-1}(\xi) \in \mathrm{H}^1(\Omega)$, owing to the Sobolev embedding $\mathrm{H}^1(\Omega) \subset L^4(\Omega)$, the last term in (4.3) is still well defined if $\mathrm{div}\boldsymbol{\tau} \in L^{4/3}(\Omega)$. This implies that $\boldsymbol{\tau} \cdot \boldsymbol{\nu} \in \mathrm{H}^{-1/2}(\Gamma)$ for all $\boldsymbol{\tau} \in \mathrm{H}(\mathrm{div}_{4/3}, \Omega)$, and as a result, the right-hand side of the first equation of (1.1) is well defined. Moreover, it readily follows that there exists $c(\Omega) > 0$ depending on $|\Omega|$ such that

$$|\, \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}, \xi \rangle_\Gamma \,| \le c(\Omega) \|\boldsymbol{\tau}\|_{\mathrm{H}(\mathrm{div}_{4/3},\Omega)} \|\xi\|_{1/2,\Gamma}, \qquad \forall\, \boldsymbol{\tau} \in \mathrm{H}(\mathrm{div}_{4/3}, \Omega),\ \forall\, \xi \in \mathrm{H}^{1/2}(\Gamma).$$

As a consequence of the latter and the Hölder inequality (2.1), it readily follows that F and G are bounded:

$$\begin{aligned} |F(\boldsymbol{\tau})| = |\, \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}, \theta_D \rangle_\Gamma \,| &\le c(\Omega) \|\theta_D\|_{1/2,\Gamma} \|\boldsymbol{\tau}\|_{\mathrm{H}(\mathrm{div}_{4/3},\Omega)} && \forall\, \boldsymbol{\tau} \in \mathrm{H}(\mathrm{div}_{4/3}, \Omega), \\ |G(v)| &\le \|g\|_{L^{4/3}(\Omega)} \|v\|_{L^4(\Omega)} && \forall\, v \in L^4(\Omega). \end{aligned} \tag{4.4}$$

Let us observe also that, owing to the Sobolev inequality (2.3), the bilinear form d is bounded:

$$|d(\boldsymbol{\tau}, \psi)| \le C_{Sob} \|\mathbf{v}\|_{1,\Omega} \|\boldsymbol{\tau}\|_{\mathrm{H}(\mathrm{div}_{4/3},\Omega)} \|\psi\|_{L^4(\Omega)} \quad \forall\, (\boldsymbol{\tau}, \psi) \in \mathrm{H}(\mathrm{div}_{4/3}, \Omega) \times L^4(\Omega). \tag{4.5}$$

The following result establishes the well-posedness of (4.2).

THEOREM 4.1. *Let $\Omega \subseteq \mathbb{R}^n$, $n = 2, 3$, be a bounded domain with Lipschitz boundary Γ. Assume that*

$$\frac{C_{Sob}(\beta^2 + 4\beta + 2)}{\beta^2} \|\mathbf{v}\|_{1,\Omega} \le \frac{1}{2} \tag{4.6}$$

with $\beta > 0$ being the constant of the inf-sup condition (2.6) with $p = 4/3$ and $q = 4$ and $C_{Sob} > 0$ the constant in (2.3). Then there exists a unique solution $(\boldsymbol{\sigma}, \theta)$ to (4.2) with $(\boldsymbol{\sigma}, \theta) \in \mathrm{H}(\mathrm{div}_{4/3}, \Omega) \times L^4(\Omega)$. In addition, there exists $C > 0$ independent of the solution such that

$$\|\boldsymbol{\sigma}\|_{\mathrm{H}(\mathrm{div}_{4/3},\Omega)} + \|\theta\|_{L^4(\Omega)} \le C(\|\theta_D\|_{1/2,\Gamma} + \|g\|_{0,\Omega}). \tag{4.7}$$

Proof. Let us first define the global bilinear form given by

$$\begin{aligned} &A : [\mathrm{H}(\mathrm{div}_{4/3}, \Omega) \times L^4(\Omega)] \times [\mathrm{H}(\mathrm{div}_{4/3}, \Omega) \times L^4(\Omega)] \to \mathbb{R}, \\ &A((\boldsymbol{\zeta}, z), (\boldsymbol{\tau}, v)) = a(\boldsymbol{\zeta}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, z) + b(\boldsymbol{\zeta}, v) + d(\boldsymbol{\zeta}, v), \end{aligned} \tag{4.8}$$

$\forall\, (\boldsymbol{\zeta}, z), (\boldsymbol{\tau}, v) \in \mathrm{H}(\mathrm{div}_{4/3}, \Omega) \times L^4(\Omega)$. In what follows we prove that A satisfies the estimates

$$S_1 := \sup_{(\boldsymbol{\tau},v) \in [\mathrm{H}(\mathrm{div}_{4/3},\Omega) \times L^4(\Omega)] \setminus \{\mathbf{0}\}} \frac{A((\boldsymbol{\zeta}, z), (\boldsymbol{\tau}, v))}{\|(\boldsymbol{\tau}, v)\|} \ge \gamma \|(\boldsymbol{\zeta}, z)\| \qquad \forall (\boldsymbol{\zeta}, z) \in \mathrm{H}(\mathrm{div}_{4/3}, \Omega) \times L^4(\Omega), \tag{4.9}$$

$$S_2 := \sup_{(\boldsymbol{\tau},v) \in \mathrm{H}(\mathrm{div}_{4/3},\Omega) \times L^4(\Omega)} A((\boldsymbol{\tau}, v), (\boldsymbol{\zeta}, z)) > 0 \qquad \forall\, (\boldsymbol{\zeta}, z) \in \mathrm{H}(\mathrm{div}_{4/3}, \Omega) \times L^4(\Omega) \setminus \{\mathbf{0}\}, \tag{4.10}$$

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and apply the Banach-Nečas-Babuška theorem (cf. [14, Theorem 2.6]) to conclude the desired result. We start with the verification of (4.9).

Let us first recall that the bilinear forms a and b are bounded with constants $\|a\| = 1$ $\|b\| = 1$, respectively. Then, since the bilinear form b satisfies the inf-sup condition (2.6) and a satisfies (2.11) with $\alpha = 1$, by applying [14, Proposition 2.36] it is easy to see that

(4.11)
$$\tilde{S}_1 := \sup_{(\boldsymbol{\tau},v)\in[\mathrm{H}(\mathrm{div}_{4/3},\Omega)\times L^4(\Omega)]\setminus\{\mathbf{0}\}} \frac{a(\boldsymbol{\zeta},\boldsymbol{\tau}) + b(\boldsymbol{\tau},z) + b(\boldsymbol{\zeta},v)}{\|(\boldsymbol{\tau},v)\|} \geq \frac{\beta^2}{\beta^2+4\beta+2}\|(\boldsymbol{\zeta},z)\|.$$

Moreover, thanks to the continuity of d (cf. (4.5)) we readily obtain

$$\begin{aligned}\hat{S}_1 := &\sup_{(\boldsymbol{\tau},v)\in[\mathrm{H}(\mathrm{div}_{4/3},\Omega)\times L^4(\Omega)]\setminus\{\mathbf{0}\}} \frac{|d(\boldsymbol{\zeta},v)|}{\|(\boldsymbol{\tau},v)\|} \leq C_{Sob}\|\mathbf{v}\|_{1,\Omega}\|\boldsymbol{\zeta}\|_{\mathrm{H}(\mathrm{div}_{4/3},\Omega)}\\ \leq &C_{Sob}\|\mathbf{v}\|_{1,\Omega}\|(\boldsymbol{\zeta},z)\|.\end{aligned}$$

According, to the above, it follows that

$$S_1 \geq \tilde{S}_1 - \hat{S}_1 \geq \left(\frac{\beta^2}{\beta^2+4\beta+2} - C_{Sob}\|\mathbf{v}\|_{1,\Omega}\right)\|(\boldsymbol{\zeta},z)\|,$$

which together with assumption (4.6) clearly implies (4.9) with $\gamma := \frac{\beta^2}{2(\beta^2+4\beta+2)}$.

Next, for (4.10) we let $(\boldsymbol{\zeta},z) \in [\mathrm{H}(\mathrm{div}_{4/3},\Omega)\times L^4(\Omega)]\setminus\{\mathbf{0}\}$ and observe that

$$S_2 \geq \bar{S}_2 := \sup_{(\boldsymbol{\tau},v)\in[\mathrm{H}(\mathrm{div}_{4/3},\Omega)\times L^4(\Omega)]\setminus\{\mathbf{0}\}} \frac{A((\boldsymbol{\tau},v),(\boldsymbol{\zeta},z))}{\|(\boldsymbol{\tau},v)\|}. \tag{4.12}$$

In turn, since $a(\cdot,\cdot)$ is symmetric, the estimate (4.11) implies

(4.13)
$$\tilde{S}_2 := \sup_{(\boldsymbol{\tau},v)\in[\mathrm{H}(\mathrm{div}_{4/3},\Omega)\times L^4(\Omega)]\setminus\{\mathbf{0}\}} \frac{a(\boldsymbol{\tau},\boldsymbol{\zeta}) + b(\boldsymbol{\zeta},v) + b(\boldsymbol{\tau},z)}{\|(\boldsymbol{\tau},v)\|} \geq \frac{\beta^2}{\beta^2+4\beta+2}\|(\boldsymbol{\zeta},z)\|,$$

and the continuity of d yields

$$\begin{aligned}\hat{S}_1 := &\sup_{(\boldsymbol{\tau},v)\in[\mathrm{H}(\mathrm{div}_{4/3},\Omega)\times L^4(\Omega)]\setminus\{\mathbf{0}\}} \frac{d(\boldsymbol{\tau},z)}{\|(\boldsymbol{\tau},v)\|} \leq C_{Sob}\|\mathbf{v}\|_{1,\Omega}\|z\|_{L^4(\Omega)}\\ \leq &C_{Sob}\|\mathbf{v}\|_{1,\Omega}\|(\boldsymbol{\zeta},z)\|.\end{aligned} \tag{4.14}$$

Therefore, from (4.12), (4.13), and (4.14) we easily obtain

$$S_2 \geq \bar{S}_2 \geq \tilde{S}_2 - \hat{S}_2 \geq \left(\frac{\beta^2}{\beta^2+4\beta+2} - C_{Sob}\|\mathbf{v}\|_{1,\Omega}\right)\|(\boldsymbol{\zeta},z)\|,$$

which combined with (4.6) implies (4.10). We conclude the proof by observing that the estimate (4.7) follows straightforwardly from (4.4), (4.9), and the fact that the embedding $L^2(\Omega) \to L^{4/3}(\Omega)$ is continuous. ∎

REMARK 4.2. As mentioned before, to obtain the well-posedness of problem (4.2) we assume that $\mathbf{v}$ is sufficiently small in the H^1-norm (see (4.6) and (4.16) below for the discrete problem). The latter, which corresponds to a preliminary result for the convection-diffusion equation, prevents a case when advection is dominant in relative comparison to diffusion. However, we continue investigating the possibility of getting rid of this strong assumption. We also notice that a similar result can be obtained by applying the theory developed in [9].

4.2. Finite element discretization of the convection-diffusion problem. Let $\mathrm{H}_h \subseteq \mathrm{H}(\mathrm{div}_{4/3}, \Omega)$ and $\mathrm{Q}_h \subseteq L^4(\Omega)$ be the finite element spaces defined in (3.2), that is,

$$\begin{aligned} \mathrm{H}_h &:= \{ \boldsymbol{\tau}_h \in \mathrm{H} \,:\, \boldsymbol{\tau}_h|_T \in \mathrm{RT}_k(T), \quad \forall\, T \in \mathcal{T}_h \}, \\ \mathrm{Q}_h &:= \{ v_h \in \mathrm{Q} \,:\, v_h|_T \in \mathbb{P}_k(T), \quad \forall\, T \in \mathcal{T}_h \}, \end{aligned}$$

where $\mathcal{T}_h$ is a regular mesh. Then the Galerkin scheme of (4.2) reads: Find a solution $(\boldsymbol{\sigma}_h, \theta_h) \in \mathrm{H}_h \times \mathrm{Q}_h$ such that

$$\begin{aligned} a(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, \theta_h) &= F(\boldsymbol{\tau}_h), \\ b(\boldsymbol{\sigma}_h, \psi_h) + d(\boldsymbol{\sigma}_h, \psi_h) &= G(\psi_h), \end{aligned} \tag{4.15}$$

for all $(\boldsymbol{\tau}_h, \psi_h) \in \mathrm{H}_h \times \mathrm{Q}_h$.

The following theorem establishes the well-posedness of the Galerkin scheme (4.15) and the corresponding a priori error estimate.

THEOREM 4.3. *Let $\Omega \subseteq \mathbb{R}^n$, $n = 2, 3$, be a bounded polyhedral domain with Lipschitz boundary Γ. Assume that*

$$\frac{C_{Sob}({\beta^*}^2 + 4\beta^* + 2)}{{\beta^*}^2} \|\mathbf{v}\|_{1,\Omega} \leq \frac{1}{2} \tag{4.16}$$

with $\beta^* > 0$ being the constant of the inf-sup condition (3.10) with $p = 4/3$ and $q = 4$. Then there exists a unique solution $(\boldsymbol{\sigma}_h, \theta_h) \in \mathrm{H}_h \times \mathrm{Q}_h$ to (4.15). In addition, there exists C_1, $C_2 > 0$ independent of h such that

$$\|\boldsymbol{\sigma}_h\|_{\mathrm{H}(\mathrm{div}_{4/3},\Omega)} + \|\theta_h\|_{L^4(\Omega)} \leq C_1 (\|\theta_D\|_{1/2,\Gamma} + \|g\|_{0,\Omega}) \tag{4.17}$$

and

$$\begin{aligned} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathrm{H}(\mathrm{div}_{4/3},\Omega)} + \|\theta - \theta_h\|_{L^4(\Omega)} \leq C_2 \Big\{ &\inf_{\boldsymbol{\tau}_h \in \mathrm{H}_h} \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathrm{H}(\mathrm{div}_{4/3},\Omega)} \\ &+ \inf_{\psi_h \in \mathrm{Q}_h} \|\theta - \psi_h\|_{L^4(\Omega)} \Big\}. \end{aligned} \tag{4.18}$$

Proof. Since a and b satisfy (3.17) and (3.10), respectively, with constants $\alpha^* = 1$ and $\beta^* > 0$, by proceeding analogously as in the proof of Theorem 4.1, it can be proved that the global bilinear form A (cf. (4.8)) satisfies the discrete version of (4.9) with the constant $\gamma^* = \frac{{\beta^*}^2}{2({\beta^*}^2 + 4\beta^* + 2)}$, that is,

$$\sup_{(\boldsymbol{\tau},v) \in [\mathrm{H}_h \times \mathrm{Q}_h] \setminus \{\mathbf{0}\}} \frac{A((\boldsymbol{\zeta}, z), (\boldsymbol{\tau}, v))}{\|(\boldsymbol{\tau}, v)\|} \geq \gamma^* \|(\boldsymbol{\zeta}, z)\| \qquad \forall\, (\boldsymbol{\zeta}, z) \in \mathrm{H}_h \times \mathrm{Q}_h, \tag{4.19}$$

Then the existence and uniqueness of solution of problem (4.15) follow from a direct application of the discrete version of the Banach-Nečas-Babuška theorem (cf. [14, Theorem 2.6]). In turn, the estimate (4.17) follows from (4.4), (4.19), and the fact that the embedding $L^2(\Omega) \to L^{4/3}(\Omega)$ is continuous. Finally, it is not difficult to see that the a priori error estimate is a direct consequence of (4.19), the continuity of a, b, and d, and [14, Lemma 2.28]. ∎

The following theorem provides the theoretical rate of convergence of the Galerkin scheme (4.15), under suitable regularity assumptions on the exact solution.

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THEOREM 4.4. *Let $\Omega \subseteq \mathbb{R}^n$, $n = 2, 3$, be a bounded polyhedral domain with Lipschitz boundary Γ. Assume that*

$$C_{Sob} \max\left\{ \frac{(\beta^2 + 4\beta + 2)}{\beta^2}, \frac{({\beta^*}^2 + 4\beta^* + 2)}{{\beta^*}^2} \right\} \|\mathbf{v}\|_{1,\Omega} \leq \frac{1}{2}$$

with $\beta > 0$ and $\beta^* > 0$ being the constant of the inf-sup conditions (2.6) and (3.10), respectively, with $p = 4/3$ and $q = 4$. Let $(\boldsymbol{\sigma}, \theta) \in \mathrm{H}(\mathrm{div}_{4/3}, \Omega) \times L^4(\Omega)$ and $(\boldsymbol{\sigma}_h, \theta_h) \in \mathrm{H}_h \times \mathrm{Q}_h$ be the unique solutions of (4.2) and (4.15), respectively. Assume that $\boldsymbol{\sigma} \in [\mathrm{H}^{l+1}(\Omega)]^n$, $\mathrm{div}\boldsymbol{\sigma} \in W^{l+1,4/3}(\Omega)$, and $\theta \in W^{l+1,4}(\Omega)$, with $0 \leq l \leq k$. Then there exists $C > 0$ independent of h such that

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathrm{H}(\mathrm{div}_{4/3},\Omega)} + \|\theta - \theta_h\|_{L^4(\Omega)} \leq C h^{l+1} \Big\{ \|\boldsymbol{\sigma}\|_{l+1,\Omega} + \|\mathrm{div}\boldsymbol{\sigma}\|_{W^{l+1,4/3}(\Omega)} + \|\theta\|_{W^{l+1,4}(\Omega)} \Big\}.$$

Proof. The proof follows from the Céa estimate (4.18), (3.8), and (3.20). ∎

REMARK 4.5. We end this section by observing that, instead of introducing the gradient of θ as a further unknown to derive the saddle-point problem (4.2), we could have introduced $\tilde{\boldsymbol{\sigma}} = \nabla\theta - \mathbf{v}\theta$ as an additional unknown to obtain the mixed problem in conservative form: Find $(\tilde{\boldsymbol{\sigma}}, \theta) \in \mathrm{H}(\mathrm{div}_{4/3}, \Omega) \times L^4(\Omega)$ such that

$$\begin{aligned} a(\tilde{\boldsymbol{\sigma}}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \theta) - d(\boldsymbol{\tau}, \theta) &= F(\boldsymbol{\tau}), \\ b(\tilde{\boldsymbol{\sigma}}, \psi) &= G(\psi), \end{aligned} \tag{4.20}$$

for all $(\boldsymbol{\tau}, \psi) \in \mathrm{H}(\mathrm{div}_{4/3}, \Omega) \times L^4(\Omega)$, where the bilinear forms a, b, and d, and the functionals F and G are defined exactly as above. Then it is not difficult to realize that the same arguments utilized above can be applied to obtain the well-posedness of (4.20) and of its corresponding Galerkin approximation.

5. Numerical results. In this section we corroborate numerically the theory developed for problem (1.1) as applied to the convection-diffusion problem (1.2). More precisely, in what follows we present three examples illustrating the performance of the Galerkin scheme (4.15) on a set of regular triangulations. Our implementation is based on a *FreeFem++* code (see [19]) in conjunction with the direct linear solver UMFPACK (see [12]).

We now introduce some additional notations. The individual errors are denoted by:

$$\mathsf{e}(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathrm{H}(\mathrm{div}_{4/3},\Omega)} \quad \text{and} \quad \mathsf{e}(\theta) := \|\theta - \theta_h\|_{L^4(\Omega)}.$$

Also, we let $r(\boldsymbol{\sigma})$ and $r(\theta)$ be the experimental rates of convergence given by

$$r(\boldsymbol{\sigma}) := \frac{\log(\mathsf{e}(\boldsymbol{\sigma})/\mathsf{e}'(\boldsymbol{\sigma}))}{\log(h/h')} \quad \text{and} \quad r(\theta) := \frac{\log(\mathsf{e}(\theta)/\mathsf{e}'(\theta))}{\log(h/h')},$$

where h and h' denote two consecutive mesh sizes with errors e and e'.

EXAMPLE 5.1. In Example 5.1 we verify the theory for the two dimensional case. To that end, we choose the domain $\Omega := (0,1)^2$, the vector field $\mathbf{v}(x_1, x_2) := (e^{x_1}, e^{x_2})^t$ and take g and θ_D so that the exact solution is given by

$$\boldsymbol{\sigma}(x_1, x_2) := \begin{bmatrix} 2x_1 \sin(\pi x_2) \\ \pi x_1^2 \cos(\pi x_2) \end{bmatrix}, \qquad \theta(x_1, x_2) := x_1^2 \sin(\pi x_2). \tag{5.1}$$

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TABLE 5.1

EXAMPLE 5.1: *Degrees of freedom, mesh sizes, errors, rates of convergence for the* RT_0-$\mathbb{P}_0$ *(top part) and* RT_1-$\mathbb{P}_1$ *(bottom part) approximations of the convection-diffusion problem* (4.2) *in 2D .*

N	h	$\mathsf{e}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$\mathsf{e}(\theta)$	$r(\theta)$
1512	0.1074	0.2124	–	0.0287	–
6124	0.0501	0.1042	0.9348	0.0137	0.9691
24273	0.0265	0.0516	1.1012	0.0069	1.0789
98206	0.0131	0.0257	0.9878	0.0034	0.9887
387402	0.0075	0.0128	1.2509	0.0017	1.2793
1541734	0.0039	0.0065	1.0506	0.0009	1.0384

N	h	$\mathsf{e}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$\mathsf{e}(\theta)$	$r(\theta)$
4800	0.1074	5.9189E-03	–	0.7287E-03	–
19520	0.0501	1.4383E-03	1.8570	0.1790E-03	1.8427
77520	0.0265	0.3610E-03	2.1639	0.0441E-03	2.1939
313952	0.0131	0.0882E-03	2.0013	0.0107E-03	2.0076
1239072	0.0075	0.0223E-03	2.4802	0.0027E-03	2.4576
4932320	0.0039	0.0056E-03	2.1196	0.0007E-03	2.1583

TABLE 5.2

EXAMPLE 5.2: *Degrees of freedom, mesh sizes, errors, and rates of convergence for the* RT_0-$\mathbb{P}_0$ *approximation of the convection-diffusion problem* (4.2) *in 2D with a not essentially bounded vector field* $\mathbf{v}$ *.*

N	h	$\mathsf{e}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$\mathsf{e}(\theta)$	$r(\theta)$
725	0.1205	0.0723	–	0.0087	–
2840	0.0627	0.0365	1.0467	0.0048	0.9170
11375	0.0318	0.0180	1.0389	0.0024	1.0332
45105	0.0181	0.0090	1.2345	0.0012	1.2360
178965	0.0095	0.0045	1.0842	0.0006	1.0812
711585	0.0056	0.0023	1.2814	0.0003	1.2910

EXAMPLE 5.2. Next in Example 5.2 we explore the performance of our mixed method under the presence of a not essentially bounded vector field $\mathbf{v}$. To that end we consider the domain $\Omega := \{(x_1,x_2)\in\mathbb{R}^2 : \|(x_1,x_2)\|_2 < \frac{1}{2}\}$ and define the function

$$\phi(x_1,x_2) := \ln(|\ln(\|(x_1,x_2)\|_2)|), \qquad \forall (x_1,x_2)\in\Omega,$$

where $\|\cdot\|_2$ denotes the Euclidean norm in $\mathbb{R}^2$. It is not difficult to see that $\phi \in \mathrm{H}^1(\Omega)$ and $\phi \notin L^\infty(\Omega)$. Then we set $\mathbf{v}(x_1,x_2) := \phi(x_1,x_2)(1,1)^t$, for all $(x_1,x_2)\in\Omega$ and compute the data g and θ_D with the functions defined in (5.1).

EXAMPLE 5.3. Finally, in Example 5.3 we assess the capability of a 3D implementation of the Galerkin scheme (4.15). Here, we choose the domain $\Omega := (0,1)^3$, the vector field $\mathbf{v}(x_1,x_2,x_3) := (x_1^2, x_2^2, 0)^t$ and take g and θ_D so that the exact solution is given by

$$\begin{aligned}\boldsymbol{\sigma}(x_1,x_2,x_3) &:= \begin{bmatrix} x_2(x_3+e^{x_3+x_1x_2}) \\ x_1(x_3+e^{x_3+x_1x_2}) \\ e^{x_3+x_1x_2}+x_1x_2 \end{bmatrix} e^{2x_1+x_2}\,,\\ \theta(x_1,x_2,x_3) &:= e^{x_3+x_1x_2}+x_1x_2x_3\,.\end{aligned}$$

In Table 5.1, we summarize the convergence history for Example 5.1 considering a sequence of regular triangulations. We observe there that the rates of convergence $O(h)$ (when

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TABLE 5.3
EXAMPLE 5.3: *Degrees of freedom, mesh sizes, errors, and rates of convergence for the* RT_0-P_0 *approximation of the convection-diffusion problem* (4.2) *in 3D* .

N	h	$\mathsf{e}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$\mathsf{e}(\theta)$	$r(\theta)$
168	0.7071	1.2336	–	0.8134	–
1248	0.3536	0.6276	0.9749	0.4231	0.9428
9600	0.1768	0.3154	0.9928	0.2137	0.9854
75264	0.0884	0.1579	0.9980	0.1071	0.9963
595968	0.0442	0.0790	0.9995	0.0536	0.9991

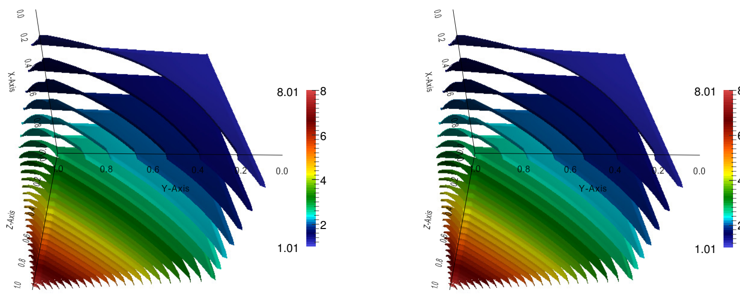

FIG. 5.1. EXAMPLE 5.3: *isosurfaces of* θ_h *(left) and* θ *(right), with* $N = 595968$.

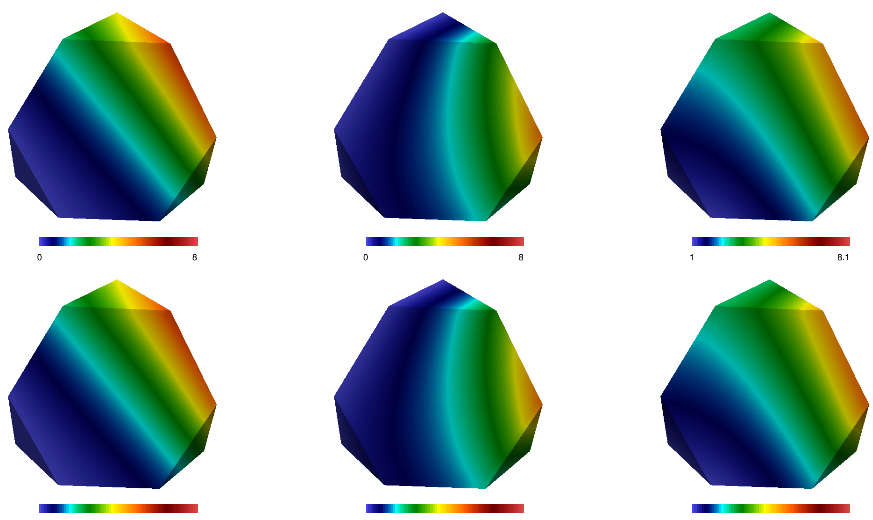

FIG. 5.2. EXAMPLE 5.3: $\boldsymbol{\sigma}_{1,h}$, $\boldsymbol{\sigma}_{2,h}$, $\boldsymbol{\sigma}_{3,h}$ *(from the left to the right, at the top) and* $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$, $\boldsymbol{\sigma}_3$ *(from the left to the right, at the bottom) with* $N = 595968$.

$k = 0$) and $O(h^2)$ (when $k = 1$) predicted by Theorem 4.4 are attained in all the cases. Similar results can be seen in Table 5.2 for the example with a not essentially bounded vector field $\mathbf{v}$ and in Table 5.3 for the 3D case. Next, in Figures 5.1 and 5.2 we provide the graphics of the approximate (with RT_0-$\mathbb{P}_0$) and exact solutions of Example 5.3. In Figure 5.1 we display the isosurface of θ_h (to the left), and we compare it with its exact counterpart (to the right). In addition, in Figure 5.2 we display the components of the vector field $\boldsymbol{\sigma}_h$ (top) and we compare them with their exact counterpart (bottom). Here, we display the section of the cube below the plane $x_1 - x_2 + x_3 = 0.5$. All the graphics were computed with $N = 595968$ degrees of freedom. We observe there that the mixed finite element method provides very accurate approximations to the unknowns. In addition, we notice that the choice of $\mathbf{v}$ in the three cases leads to a good behavior of the numerical method. It is pertinent to mention here that the actual influence of assumption (4.16) on the performance of the numerical approximation of (4.2) in the examples is not analyzed in this work since it is outside of the original scope of this paper and remains an open problem to be addressed in the future. However, there is numerical evidence showing that when having non-symmetric structures as the one presented in (4.2), the associate global matrix of the system becomes ill-posed as $\|d\|$ is too large (see [8, Section 7, Example 1]).

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